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## A Note on Deriving a Result in Decision Theory

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<p>This note provides an alternate derivation of a well-known problem in decision theory, which deals with finding the minimum of a certain expectation, i.e., <math>E X - c </math>. Textbooks generally present this problem as an exercise to verify that the median does, in fact, yield a minimum. The derivation presented here directly finds the minimum, and yields asymptotic properties of the expectation.</p>					
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## A NOTE ON DERIVING A RESULT IN DECISION THEORY

### INTRODUCTION

A number of textbooks in mathematical statistics [see, e.g., Ref. 1] include an exercise to verify that if  $X$  is a continuous random variable with finite expectation  $E(X)$  and density function  $f(x)$ , then  $E|X - c|$  is minimized for  $c = m$ , where  $m$  is the median of  $f$ . This result is used in statistical decision theory [see, e.g., Ref. 2].

The usual way authors show this is to first show that

$$E|X - c| = E|X - m| + 2 \int_m^c (c - x) f(x) dx .$$

Since the term containing the integral in the above equation is nonnegative, one concludes that  $E|X - m| \leq E|X - c|$ , and thus  $E|X - c|$  is minimized at  $c = m$ .

### MINIMIZING THE EXPECTED VALUE

The reader might be interested in knowing how one determines beforehand that the median is a plausible candidate for the global minimum. The exercise above asks to show that the median does, in fact, yield a minimum. However, one may not see readily why it is reasonable to choose the median in the first place. This note addresses this issue. We will show directly that the median is the required value which yields the global minimum for  $E|X - c|$ . Our derivation will also show the asymptotic nature of this expectation.

Defining  $G(c) \equiv E|X - c|$ , we compute  $G'(c)$  and show that it vanishes at  $c = m$ . We have,

$$\begin{aligned} G(c) &= \int_{-\infty}^{\infty} |x - c| f(x) dx \\ &= \int_{-\infty}^c (c - x) f(x) dx + \int_c^{\infty} (x - c) f(x) dx \\ &= c \left[ \int_{-\infty}^c f(x) dx - \int_c^{\infty} f(x) dx \right] \\ &\quad + \int_c^{\infty} xf(x) dx - \int_{-\infty}^c xf(x) dx . \end{aligned}$$

Now differentiating  $G(c)$  with respect to  $c$ , we arrive at

$$\begin{aligned} G'(c) &= \int_{-\infty}^c f(x) dx - \int_c^{\infty} f(x) dx \\ &= 2F(c) - 1 , \end{aligned}$$

where  $F(x)$  is the cumulative density function. Finally, we see that  $G'(c) = 0$  if and only if  $c = m$ . Thus we have found a possible local minimum at  $c = m$ .

Now it is easy to show that we have the global minimum at the median. Suppose  $c > m$ . Then  $G'(c) \geq 0$ . Further, for  $c < m$ , we have  $G'(c) \leq 0$ . Thus, we conclude that  $G(c)$  assumes an absolute or global minimum at  $c = m$ .

### ASYMPTOTES FOR THE EXPECTED VALUE

It is interesting to note that as  $c \rightarrow \pm \infty$ , then  $G'(c) \rightarrow \pm 1$ . Also, we note that for  $c \rightarrow \infty$ ,

$$G(c) \sim c - E(X) ,$$

and for  $c \rightarrow -\infty$ ,

$$G(c) \sim -c + E(X) ,$$

where " $\sim$ " means "asymptotically equal to". Equivalently, we can write

$$\lim_{c \rightarrow \pm \infty} \frac{E(X) \pm G(c)}{c} = 1 .$$

Further, we remark that  $G(c)$  always lies above the asymptotes.

### REFERENCES

1. Cramér, Harold (1946), *Mathematical Methods of Statistics*, Princeton University Press, Princeton, N.J.
2. DeGroot, Morris H. (1970), *Optimal Statistical Decisions*, McGraw-Hill, New York.